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On Spherical Cycles in the Complement to Complex Hypersurfaces

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It is known due to S.Yu. Nemirovski, that for $n \geqslant 3$ and generic hypersurface $V \subset \mathbb{C}^n$ of degree $d \geqslant 3$ there exists a sum of the Whitney spheres homotopic to an embedded sphere, which represents a nontrivial homological class of the homology group $H_n(\mathbb{C}^n \setminus V)$. We discuss whether a linear combination of the Whitney spheres can be represented as an embedded sphere.

Keywords: homology group, embedding, Whitney sphere.

Introduction

The structure of n-dimensional cycles in the complement to an algebraic hypersurface V in $\mathbb{C}^n$ is interesting from different points of view. For example, such cycles $\gamma \in Z_n(\mathbb{C}^n \setminus V)$ are the integration sets for multidimensional integrals (residues)

$$\int_\gamma \frac{P(z)}{Q(z)}\, dz_1 \wedge \ldots \wedge dz_n$$

of rational functions with poles along the hypersurface $V = \{z \in \mathbb{C}^n : Q(z) = 0\}$ and cycles $\gamma \in Z_n(\mathbb{C}^n \setminus V)$ (see [1], [2], [3]). The results of Leray (see [4, § 19]) lead to the following.

Theorem 1. *For a smooth hypersurface $V \subset \mathbb{C}^n$ there exists an isomorphism*

$$H_n(\mathbb{C}^n \setminus V) \simeq H_{n-1}(V).$$

This isomorphism is given by the Leray coboundary operator (coboundary of Leray).

We recall the key points of its construction: Let γ be a cycle on the hypersurface V. For each point $z \in \gamma$ we consider the set δ_z of end-points of geodesic segments, beginning at z and orthogonal to V. The lengths $\rho(z)$ of these segments are chosen to be sufficiently small and such that the function $\rho(z)$ is smooth.

The *coboundary* or *tube* $\delta(\gamma)$ is defined as the union $\delta(\gamma) = \bigcup\limits_{z\in\gamma} \delta_z$.

Thus, according to Theorem 1, every cycle in the complement $\gamma \in Z_n(\mathbb{C}^n \setminus V)$ is homologous to a tube over some cycle $\sigma \in Z_{n-1}(V)$ on the surface V.

At first glance, it may seem that all cycles in the complement have exclusively tubular structure, however it is easy to provide an example proving that a tube may be homologous to a sphere.

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Example 1. *There is only one nontrivial homology class in the punctured space $\mathbb{R}^3 \setminus \{0\}$. This class may be represented by both sphere S^2 and torus T^2 with the origin inside. Thus, cycles S^2 and T^2 are homologous in $\mathbb{R}^3 \setminus \{0\}$.*

The following problem was posed by A.K.Tsikh (see, for example, [5]).

Problem. *Does there exists a nontrivial n-dimensional homology class in the complement of a hypersurface in $\mathbb{C}^n$ that can be represented by an embedded sphere?*

The answer to this question was given in 2001 by S.Yu.Nemirovskii. Recall that an affine algebraic hypersurface V is called *generic* if its projective closure $\overline{V} \subset \mathbb{CP}^n$ is nonsingular and intersects the hyperplane at infinity transversally.

Theorem 2 (Nemirovskii [6])**.** *Suppose that $V \subset \mathbb{C}^n$ is a generic algebraic hypersurface of degree d. If $n \geqslant 3$ and $d \geqslant 3$, then there exists a smoothly embedded n-sphere representing a nontrivial homology class in $\mathbb{C}^n \setminus V$.*

In the case $n = 2$ or $d = 2$ such an embedding is impossible.

The Nemirovskii construction of a nontrivial cycle in $\mathbb{C}^n \setminus V$ is realised as a sum of the Whitney spheres. Our aim is to investigate whether a linear combination of the Whitney spheres can be represented as an embedded sphere.

In the first part of this paper we recall the definition of Fermat hypersurface, the homological structure of such hypersurfaces and the universal covering of their complement. We devote the second part to a brief review of the construction of the Whitney spheres. In the third part we prove the main result, a criterion for a linear combination of the Whitney spheres to be represented by an embedded sphere. The last part contains a detailed analysis of an example.

1. The Fermat Hypersurfaces and the Universal Covering of their Complement

Note that any generic hypersurface is isotopic to the Fermat hypersurface of the same degree. Since we are interested in the topology of a hypersurface and their complement, we can consider only the case when V is a Fermat hypersurface, i.e. $F = F(n, d) = \{z \in \mathbb{C}^n : \ z_1^d + \cdots + z_n^d = 1\}$. The homotopic type of the Fermat hypersurface is the wedge of spheres (see [7])

$$\underbrace{S^{n-1} \vee \cdots \vee S^{n-1}}_{(d-1)^n \text{ times}}. \tag{1}$$

The homology group $H_{n-1}(F)$ of this surface was described by Pham in [8]. He constructed the homology basis as the collection of spheres

$$s_k = s_{k_1 \dots k_n}, \qquad k = (k_1, \dots, k_n) \in \{0, \dots, d-2\}^n.$$

In notation $|\,l\,| = l_1 + \cdots + l_n$, $|k| = k_1 + \cdots + k_n$ the homology classes $e_k = [s_k] \in H_{n-1}(F, \mathbb{Z})$ satisfy the intersection formulas

$$e_k \cdot e_k = (-1)^{\frac{n(n-1)}{2}} (1 + (-1)^{n-1}), \tag{2}$$

$$e_k \cdot e_l = (-1)^{\frac{n(n-1)}{2}} (-1)^{|\,l\,| - |k|}, \text{ if } k \neq l \text{ and } k_j \leqslant l_j \leqslant k_j + 1 \text{ for } j = 1, \dots, n, \tag{3}$$

$$e_k \cdot e_l = 0, \text{ otherwise.} \tag{4}$$

It is known (see e.g. [6]) that the universal covering of the complement to the hypersurface F: $\mathcal{U}(\mathbb{C}^n \setminus F) \to \mathbb{C}^n \setminus F$, can be realized as the projection of the complex hypersurface

$$\mathcal{U}(\mathbb{C}^n \setminus F) = \{z \in \mathbb{C}^{n+1} : \ e^{z_{n+1}} + z_1^d + \cdots + z_n^d = 1\} \tag{5}$$

to the hyperplane $\mathbb{C}^n = \{z \in \mathbb{C}^{n+1} : z_{n+1} = 0\}$.

The homologies of the surfaces (5) are known also due to Pham [8]. The collection of spheres

$$s_{k_1 \ldots k_n k_{n+1}}, \quad k = (k_1, \ldots, k_n) \in \{0, \ldots, d-2\}^n, k_{n+1} \in \mathbb{Z}$$

generate the homology of $\mathcal{U}(\mathbb{C}^n \setminus F)$. The intersection formulas (2)–(4) are valid for corresponding homology classes taking into account that the dimension now is $n+1$.

2. The Whitney Spheres and their Lifts to the Universal Covering

We are now going to describe the construction of a nontrivial cycle in the complement $\mathbb{C}^n \setminus F$ which is homotopic to an embedded sphere.

Consider the fibration of the complex space $\mathbb{C}^n$ by the hypersurfaces

$$F_t = F_t(n, d) = \{z \in \mathbb{C}^n : \sum_{j=1}^{n} z_j^d + L(z_1, \ldots, z_n) = t\}, \quad t \in \mathbb{C},$$

where $L(z_1, \ldots, z_n) = \sum\limits_{j=1}^{n} \varepsilon_j z_j$ is a small linear form. It is possible to choose L in a such way that each hypersurface F_t is either generic or has one quadratic singularity in a neighborhood of which the family $\{F_t\}$, $t \in \mathbb{C}$ is biholomorphically equivalent to the family $\left\{\sum\limits_{j=1}^{n} z_j^2 = t\right\}$, $|t| < \varepsilon$. By Bezout's theorem the number N of singular fibers F_{t_j} is equal to $(d-1)^n$, i.e. coincides with the number of spheres in the wedge (1) and with the rank of the group $H_{n-1}(F)$. There is one to one correspondence between singular fibers and spherical cycles from the generic hypersurface. The fiber F_{t_j} corresponds to the cycle $s_{k_1 \ldots k_n k_{n+1}} \subset F_t$, which contracts to the singular point when the fiber F_t moves along the path from t to t_j in the space $\mathbb{C}$ of parameters t.

Now the generators of the fundamental group $\pi_n(\mathbb{C}^n \setminus F, *)$ can be constructed as follows. We take the "vanishing cycle" s_k and move it along the closed path around the generic fiber F across the corresponding singular point. In this way we get a collection

$$[w_k] = [w_{k_1 \ldots k_n}] \in \pi_n(\mathbb{C}^n \setminus F, *), \qquad k = (k_1, \ldots, k_n) \in \{0, \ldots, d-2\}^n$$

of an immersed n-spheres

$$S^n \to w_k \subset \mathbb{C}^n \setminus F \tag{6}$$

with a single transverse double self-intersection point.

As a local model in the neighborhood of the intersection point we use *the Whitney sphere* which is given by

$$\begin{array}{ccc} (x_1, \ldots, x_n, y) & \longrightarrow & ((1+iy)x_1, \ldots, (1+iy)x_n). \\ \cap & & \cap \\ S^n \times \mathbb{R} & & \mathbb{C}^n \setminus F \end{array}$$

Therefore we call the immersions (6) *the Whitney spheres* in $\mathbb{C} \setminus F$.

The lift of the Whitney spheres w_k to the universal covering $\mathcal{U}(\mathbb{C}^n \setminus F) \to \mathbb{C}^n \setminus F$ is the infinite chain of embedded n-spheres $\{\widetilde{w}_{kq}\}_{q\in\mathbb{Z}}$ such that

$$[\widetilde{w}_{kq}] \cdot [\widetilde{w}_{kp}] = \begin{cases} \pm 1, & |q-p| = 1, \\ 0, & |q-p| > 1. \end{cases}$$

Moreover the collection $\{\widetilde{w}_{kq}\}_{k\in I, q\in\mathbb{Z}}$ coincides with the set of generators of homology group for $\mathcal{U}(\mathbb{C}^n \setminus F)$, described in section 1.

The Whitney spheres are those elements from which the spherical cycle in the complement $\mathbb{C}^n \setminus F$ is constructed in [6]. More precisely, a cycle

$$W = w_{0010\ldots0} + w_{0110\ldots0} + w_{0100\ldots0} + w_{1100\ldots0} + w_{1000\ldots0} + w_{1010\ldots0} \tag{7}$$

represents a nontrivial homological class and it is homotopic to an embedded sphere.

3. A Criterion for a Linear Combination of the Whitney Spheres to be Represented by an Embedded Sphere

According to result of Nemirovskii, in the case $n \geqslant 3$, $d \geqslant 3$ there is at least one cycle in the complement $\mathbb{C}^n \setminus F$ which can be represented by an embedded sphere. We are interested in how many such cycles exist, and what linear combinations of the Whitney spheres are homotopic to an embedded sphere.

We need the following statement.

Proposition 1 (Nemirovskii [6]). *Let $f : S^n \to M^{2n}$ be a smooth map of an n-dimensional sphere into an oriented $2n$-dimensional manifold, and let $\widetilde{f} : S^n \to \mathcal{U}(M^{2n})$ be its lift to the universal covering. If $n \geqslant 3$, then f is homotopic to an embedding if and only if the intersection index*

$$[\widetilde{f}(S^n)] \cdot [\tau \circ \widetilde{f}(S^n)] = 0$$

for every nontrivial deck transformation $\tau \in \pi_1(M^{2n}, *)$.

The deck transformation $\tau \circ \widetilde{f}(S^n)$ acts as follows. If a point of base M^{2n} moves along a loop τ of the fundamental group $\pi_1(M^{2n}, *)$, then the corresponding points in the universal covering $\mathcal{U}(M^{2n})$ move from one sheet to another. Thus a traversal of the loop gives a shift of the lifting $\widetilde{f}(S^n)$ by one sheet.

Now we are able to formulate the main result. Recall our notations $|\,l\,| = l_1 + \cdots + l_n$, $|k| = k_1 + \cdots + k_n$.

Theorem 3. *Let $\{w_k\}_{k\in I}$, $I = \{0, \ldots, d-2\}^n$ be a collection of the Whitney spheres in the complement to the Fermat hypersurface $F(n,d)$, $n \geqslant 3$, $d \geqslant 3$. A cycle*

$$W = \sum_{k\in I} \lambda_k w_k, \quad \lambda_k \in \mathbb{Z}, \tag{8}$$

can be represented by an embedded in $\mathbb{C}^n \setminus F(n,d)$ sphere S^n if and only if the coefficients λ_k satisfy the following second-degree Diophantine equation

$$\sum_{k\in I} \sum_{l\in J} (-1)^{|\,l\,|-|k|}\, \lambda_k \cdot \lambda_l = 0, \tag{9}$$

where $J = \{k_1, \min(k_1+1, d-2)\} \times \cdots \times \{k_n, \min(k_n+1, d-2)\}$.

Proof. Consider a cycle $W = \sum\limits_{k\in I} \lambda_k w_k,\ \lambda_k \in \mathbb{Z}$.

If $\{\widetilde{w}_{kq}\}_{k\in I, q\in\mathbb{Z}}$ are the lifts of the Whitney spheres $\{w_k\}_{k\in I}$ to the universal covering $\mathcal{U}(\mathbb{C}^n \setminus F) \to \mathbb{C}^n \setminus F$, then lifts of the cycle W are of the form $\widetilde{W}_q = \sum\limits_{k\in I} \lambda_k \widetilde{w}_{kq},\ q \in \mathbb{Z}$.

The corresponding homology classes of the universal covering can be written as

$$E_q = \sum_{k\in I} \lambda_k e_{kq},$$

where $e_{kq} = [\widetilde{w}_{kq}]$.

In order to apply Proposition 1 we need to calculate the intersection indexes of the class E_0 with all its shifts by deck transformations that correspond to every possible loop from $\pi_1(\mathbb{C}^n \setminus F)$.

Note that every such shift moves E_0 into some E_p, where $p \in \mathbb{Z}$.

According to formulas (2)–(4), the intersection index

$$E_0 \cdot E_p = 0 \quad \text{for} \quad p \neq 0, 1.$$

Calculate the intersection indexes of E_0 with their shift E_1. Since

$$e_{k0} \cdot e_{l1} = \begin{cases} (-1)^{\left[\frac{n(n+1)}{2} - 1\right]} (-1)^{|\,l\,| - |k|}, & \text{if } k_j \leqslant l_j \leqslant k_j + 1,\ j = 1, \dots, n, \\ 0, & \text{otherwise}, \end{cases}$$

one has

$$E_0 \cdot E_1 = \left[\sum_{k\in I} \lambda_k e_{k0}\right] \cdot \left[\sum_{l\in I} \lambda_l e_{l1}\right] = \sum_{k\in I}\sum_{l\in I} \lambda_k \lambda_l e_{k0} \cdot e_{l1} =$$
$$= (-1)^{\left[\frac{n(n+1)}{2} - 1\right]} \sum_{k_1=0}^{d-2} \cdots \sum_{k_n=0}^{d-2} \sum_{l_1=k_1}^{\min(k_1+1, d-2)} \cdots \sum_{l_n=k_n}^{\min(k_n+1, d-2)} \lambda_k \lambda_l (-1)^{|\,l\,| - |k|}.$$

We therefore have

$$E_0 \cdot E_1 = 0 \quad \Longleftrightarrow \quad \sum_{k\in I} \sum_{l\in J} (-1)^{|\,l\,| - |k|} \lambda_k \cdot \lambda_l = 0,$$

where $J = \{k_1, \min(k_1 + 1, d-2)\} \times \dots \times \{k_n, \min(k_n + 1, d-2)\}$.

By Proposition 1 the cycle $W = \sum_{k\in I} \lambda_k w_k$ is spherical if and only if the coefficients λ_k satisfy equation (9). $\square$

4. Spherical Cycles in the Complement to the Cubic Hypersurface in $\mathbb{C}^3$

Consider the case $d = 3$, $n = 3$. The Fermat hypersurface is then given by

$$F(3,3) = \{z \in \mathbb{C}^3 :\ z_1^3 + z_2^3 + z_3^3 = 1\}.$$

Diophantine equation (9) looks as

$$\begin{aligned} &\lambda_{000}\lambda_{001} + \lambda_{000}\lambda_{010} + \lambda_{000}\lambda_{100} + \lambda_{000}\lambda_{111} + \lambda_{001}\lambda_{011} + \lambda_{001}\lambda_{101} + \\ &\lambda_{010}\lambda_{011} + \lambda_{010}\lambda_{110} + \lambda_{011}\lambda_{111} + \lambda_{100}\lambda_{101} + \lambda_{100}\lambda_{110} + \lambda_{101}\lambda_{111} + \\ &\lambda_{110}\lambda_{111} - \lambda_{000}\lambda_{011} - \lambda_{000}\lambda_{101} - \lambda_{000}\lambda_{110} - \lambda_{001}\lambda_{111} - \lambda_{010}\lambda_{111} - \\ &\lambda_{100}\lambda_{111} - \lambda_{000}^2 - \lambda_{001}^2 - \lambda_{010}^2 - \lambda_{100}^2 - \lambda_{011}^2 - \lambda_{101}^2 - \lambda_{110}^2 - \lambda_{111}^2 = 0. \end{aligned} \tag{10}$$

To find the solutions to this equation we change variables $\lambda = (\lambda_{000}, \lambda_{001}, \lambda_{010}, \lambda_{100}, \lambda_{011}, \lambda_{101}, \lambda_{110}, \lambda_{111})$ to variables $x = (x_1, \ldots, x_8)$ of the space $\mathbb{R}^8$ by the formula $\lambda = Ax^T$, where the column x^T is the transpose of x and a transform matrix A is

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -\sqrt{3/7} & -1 & 1 \\ 0 & 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{6} & -1/\sqrt{2} & 2/\sqrt{21} & 0 & 1 \\ 0 & -\sqrt{2/3} & 0 & \sqrt{2/3} & 0 & 2/\sqrt{21} & 0 & 1 \\ 0 & 1/\sqrt{6} & -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{2} & 2/\sqrt{21} & 0 & 1 \\ 0 & -1/\sqrt{6} & -1/\sqrt{2} & -1/\sqrt{6} & -1/\sqrt{2} & -2/\sqrt{21} & 1 & 0 \\ 0 & \sqrt{2/3} & 0 & \sqrt{2/3} & 0 & -2/\sqrt{21} & 1 & 0 \\ 0 & -1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{2} & -2/\sqrt{21} & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & \sqrt{3/7} & 1 & -1 \end{pmatrix}.$$

In new variables the equation (10) looks as

$$x_1^2 + x_2^2 + x_3^2 + 3x_4^2 + 3x_5^2 + 7x_6^2 = 0. \tag{11}$$

The solution to (11) is, obviously, $x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = 0$.

Going back to the original variables, we see that all solutions to equation (10) are parameterized as follows

$$\lambda = (-x_7 + x_8, x_8, x_8, x_8, x_7, x_7, x_7, x_7 - x_8), \qquad x_7,\ x_8 \in \mathbb{R}.$$

We conclude that equation (10) has only two linearly independent solutions

$$(\lambda_{000}, \lambda_{001}, \lambda_{010}, \lambda_{100}, \lambda_{011}, \lambda_{101}, \lambda_{110}, \lambda_{111}) = (-1, 0, 0, 0, 1, 1, 1, 1),$$
$$(\lambda_{000}, \lambda_{001}, \lambda_{010}, \lambda_{100}, \lambda_{011}, \lambda_{101}, \lambda_{110}, \lambda_{111}) = (1, 1, 1, 1, 0, 0, 0, -1),$$

and all other solutions are linear combinations of them.

The obtained solutions allow us to describe all spherical cycles that are a linear combination of the Whitney spheres.

Proposition 2. *Let $\{w_k\}_{k\in I}$, $I = \{0,1\}^3$ be a collection of the Whitney spheres in the complement of the Fermat hypersurface $F(3,3)$. A cycle*

$$W = \sum_{k\in I} \lambda_k w_k$$

can be represented by an embedded sphere if and only if $W = a\gamma_1 + b\gamma_2$, where a, $b \in \mathbb{Z}$,

$$\gamma_1 = -w_{000} + w_{011} + w_{101} + w_{110} + w_{111},$$
$$\gamma_2 = w_{000} + w_{100} + w_{010} + w_{001} - w_{111}.$$

Proof. The statement of the proposition follows immediately from the solution to equation (10) and Theorem 2. □

Notice that cycle (7) for $F(3,3)$ constructed by Nemirovskii is the sum of cycles γ_1 and γ_2 from Proposition 2

$$W = \gamma_1 + \gamma_2.$$

In the case $n > 3$ or $d > 3$ the structure of the set of solutions to (9) is more complicated. For example, this set is not a linear subspace even for $n = 4$ and $d = 3$.

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О сферических циклах в дополнении к комплексным гиперповерхностям

Наталья А. Бушуева

Известен следующий результат С.Ю. Немировского: для $n \geqslant 3$ и общей гиперповерхности $V \subset \mathbb{C}^n$ степени $d \geqslant 3$ существует сумма сфер Уитни, которая гомотопна вложенной сфере и представляет гомологически нетривиальный класс гомологий группы $H_n(\mathbb{C}^n \setminus V)$. В статье выясняется вопрос о представимости заданной линейной комбинации сфер Уитни вложенной сферой.

Ключевые слова: группа гомологий, вложение, сфера Уитни.